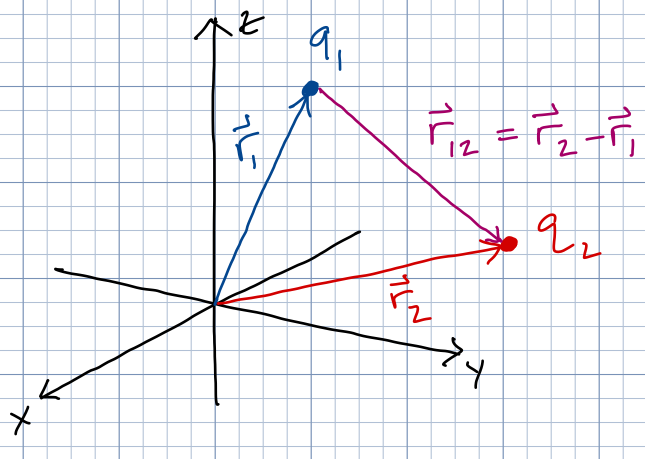
Forces Between Charges & Dipoles

- One of the 1st things you learn about point charges is Coulomb's Law



$$\vec{F}_{12} = \frac{1}{4\pi\varepsilon_0} \frac{q_1 q_2}{|\vec{r}_{12}|^2} \hat{r}_{12}$$

$$\vec{F}_{21} = -\vec{F}_{12}$$

- How would we derive this if we only knew about electrostatic potential and potential energy?

- $V(\vec{r})$ due to charge q_1 @ $\vec{r}_1$ is

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \frac{q_1}{|\vec{r} - \vec{r}_1|}$$

- If charge q_2 is @ $\vec{r}_2$ then the pair has potential energy

$$U = q_2 V(\vec{r}_2) = \frac{1}{4\pi\varepsilon_0} \frac{q_1 q_2}{|\vec{r}_2 - \vec{r}_1|}$$

- To get the force $\vec{F}_{12}$ that q_1 exerts on q_2, we take minus the gradient of U with respect to the pos. $\vec{r}_2$ of q_2. That is, how does U change if we move q_2 a little?

$$\vec{F}_{12} = -\vec{\nabla}_2 U = -\hat{x}\frac{\partial U}{\partial x_2} - \hat{y}\frac{\partial U}{\partial y_2} - \hat{z}\frac{\partial U}{\partial z_2}$$

$$|\vec{r}_{12}| = \left((x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2\right)^{1/2} \Rightarrow -\vec{\nabla}_2\left(\frac{1}{|\vec{r}_{12}|}\right) = \frac{\hat{r}_{12}}{|\vec{r}_{12}|^2}$$

- Likewise, to get $\vec{F}_{21}$ we'd take $-\vec{\nabla}_1 U$. So:

$$\vec{F}_{12} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{|\vec{r}_{12}|^2}\hat{r}_{12} \qquad \vec{F}_{21} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{|\vec{r}_{12}|^2}(-\hat{r}_{12})$$

$\hat{r}_{21} = -\hat{r}_{12}$

- This way of working out forces is really handy if we want to look @ how charges & dipoles respond to each other.

- For example, an ideal dipole $\hat{p}$ located @ $\vec{r}_1$ produces an electrostatic potential

$$V_{dip}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{\vec{p}\cdot\hat{\mathscr{r}}}{|\vec{\mathscr{r}}|^2} = \frac{1}{4\pi\varepsilon_0}\frac{\vec{p}\cdot(\vec{r}-\vec{r}_1)}{|\vec{r}-\vec{r}_1|^3}$$

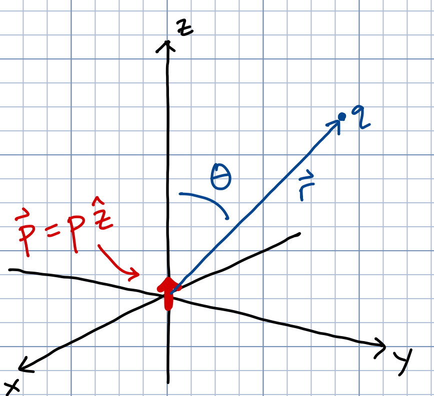
RECALL: $\dfrac{\hat{\mathscr{r}}}{\mathscr{r}^2} = \dfrac{\vec{\mathscr{r}}}{\mathscr{r}^3}$

So if a point charge q is placed @ $\vec{r}_2$, their PE is

$$U = q\,V_{dip}(\vec{r}_2) = \frac{1}{4\pi\varepsilon_0}\frac{q\,\vec{p}\cdot(\vec{r}_2-\vec{r}_1)}{|\vec{r}_2-\vec{r}_1|^3}$$

- What forces do they exert on each other? We can find them as we did above.

- To simplify the math a little, let's say the dipole is @ the origin ($\vec{r}_1 = 0$). It has a direction, $\hat{p}$, so we'll orient our axes so that $\hat{p}$ is the $\hat{z}$ direction.



$$U = \frac{qp}{4\pi\varepsilon_0}\frac{\hat{z}\cdot\hat{r}}{r^2}$$

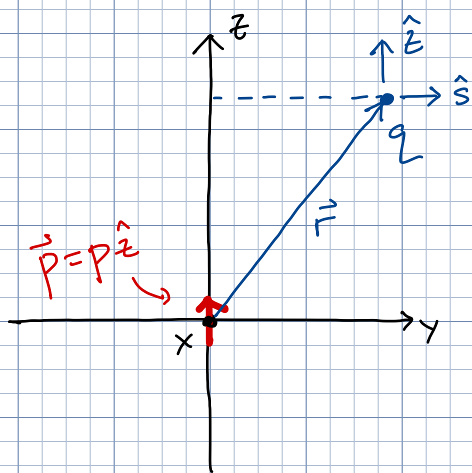
$$= \frac{qp}{4\pi\varepsilon_0}\frac{\cos\theta}{r^2}$$

SPHERICAL POLAR

$$= \frac{qp}{4\pi\varepsilon_0}\frac{z}{(s^2+z^2)^{3/2}}$$

WE'LL USE CYLINDRICAL COORDS

- The dipole singles out a direction, which we called $\hat{z}$. If we put a charge q @ $\vec{r} = s\hat{s} + z\hat{z}$ (in cylindrical coords) then the force exerted on it by the dipole $p\hat{z}$ @ the origin is:

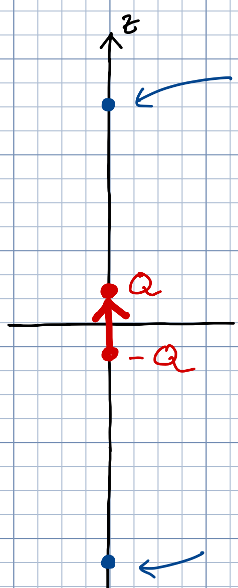


$$\vec{F} = -\vec{\nabla} U = -\frac{pq}{4\pi\varepsilon_0} \vec{\nabla}\left(\frac{s}{(s^2+z^2)^{3/2}}\right)$$

$$\hookrightarrow \vec{F} = \frac{pq}{4\pi\varepsilon_0} \times \left(\frac{3sz}{(s^2+z^2)^{5/2}}\hat{s} + 0\hat{\phi} + \frac{(2z^2-s^2)}{(s^2+z^2)^{5/2}}\hat{z}\right)$$

U depends on s & z, so $\vec{F}$ has no $\hat{\phi}$ component.

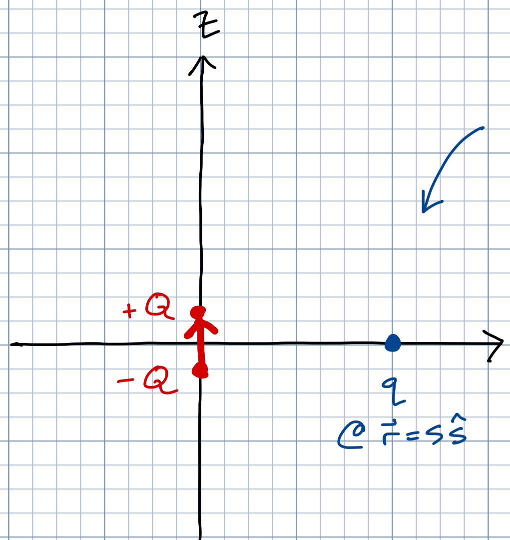
- This is very different than the Coulomb force b/t two point charges. For starters, it is $\sim \frac{1}{r^3}$ rather than $\frac{1}{r^2}$. But it also depends on the direction $\hat{r}$ relative to $\hat{p}$, which makes sense if we think of $\vec{p}$ as two charges $\pm Q$ separated in the $\hat{z}$ direction.



q @ $z\hat{z}$ $\Rightarrow \vec{F} = \frac{pq}{4\pi\varepsilon_0} \times \frac{2z^2}{(z^2)^{5/2}}\hat{z} = \frac{pq}{4\pi\varepsilon_0}\frac{\hat{z}}{z^3}$ ($z>0$!)

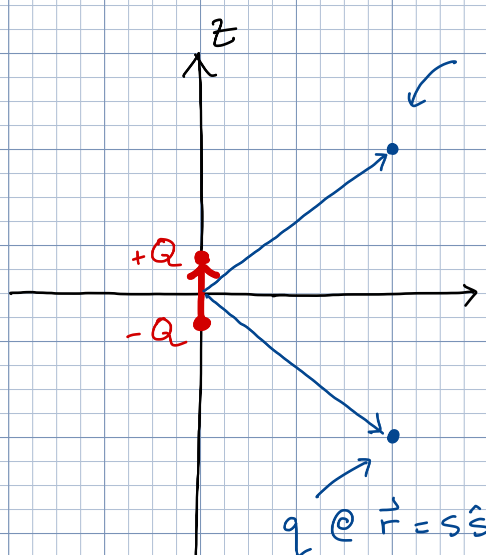
q is a little closer than $+Q$ than $-Q$, so if q is positive it is marginally repelled in the $+\hat{z}$ direction & if q is negative it is marginally attracted in the $-\hat{z}$ direction.

If q is down here ($z<0$) then $(z^2)^{5/2} = -z^5$ and the force flips direction compared to $z>0$. That makes sense: closer to $-Q$ & further from $+Q$.



q @ $\vec{r} = s\hat{s} + 0\hat{z}$ so $\vec{F} = -\dfrac{pq}{4\pi\varepsilon_0}\dfrac{\hat{z}}{s^3}$

Again, if $q > 0$ then $+Q$ repels it & $-Q$ attracts it. The net effect is a force in the $-\hat{z}$ dir.



q @ $\vec{r} = s\hat{s} + z\hat{z}$ w/ $z > 0$

$$\vec{F} = \frac{qp}{4\pi\varepsilon_0} \times \left(\frac{3sz}{(s^2+z^2)^{5/2}}\hat{s} + \frac{(2z^2 - s^2)}{(s^2+z^2)^{5/2}}\hat{z} \right)$$

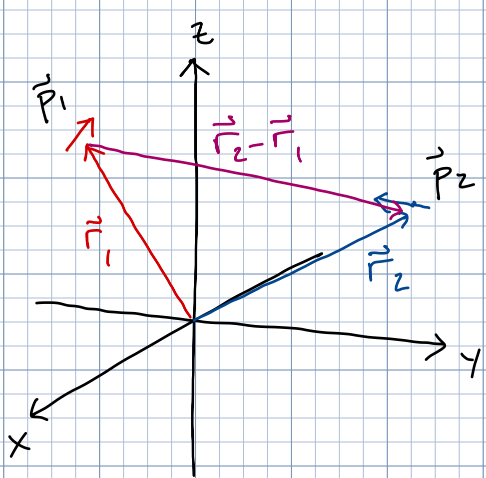
The components make sense when we think of q being closer to $+Q$ for $z > 0$ & $-Q$ for $z < 0$.

q @ $\vec{r} = s\hat{s} + z\hat{z}$ w/ $z < 0$

- So once we consider a point charge q interacting with a dipole $\vec{p}$, we get a qualitatively different sort of force than we did for two point charges.
- It is weaker in the sense that it falls off like $1/r^3$ rather than $1/r^2$, and it depends on the direction of the separation vector between $\vec{p}$ & q.
- What about two dipoles?

- Dipole-dipole forces are important b/c most non-conducting matter is essentially made of little neutral chunks that behave like dipoles. These forces are important in Chem & Bio.

- We can work out the PE for a pair of dipoles separated by $\vec{\mathscr{r}}$, as we did above. But it's more involved, so lets just skip to U:



$$U = \frac{1}{4\pi\varepsilon_0}\frac{1}{\mathscr{r}^3}\left(\vec{p}_1\cdot\vec{p}_2 - 3(\vec{p}_1\cdot\hat{\mathscr{r}})(\vec{p}_2\cdot\hat{\mathscr{r}})\right)$$

with $\vec{\mathscr{r}} = \vec{r}_2 - \vec{r}_1$ & $\hat{\mathscr{r}} = \dfrac{\vec{r}_2 - \vec{r}_1}{|\vec{r}_2 - \vec{r}_1|}$

Much more complicated than previous example! Depends on magnitudes of dipoles, their relative directions, and their directions relative to their separation vector.

- To simplify our calculation lets set up our coords so that $\vec{p}_1$ is @ the origin and the second dipole $\vec{p}_2$ is @ $\vec{r}_2 = \vec{r}$:

$$U = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^3}\left(\vec{p}_1\cdot\vec{p}_2 - 3(\vec{p}_1\cdot\hat{r})(\vec{p}_2\cdot\hat{r})\right)$$

- Then the force on $\vec{p}_2$ due to $\vec{p}_1$ is:

$$\vec{F}_{12} = -\vec{\nabla}U = -\frac{1}{4\pi\varepsilon_0}\vec{\nabla}\left(\frac{1}{r^3}\left(\vec{p}_1\cdot\vec{p}_2 - 3(\vec{p}_1\cdot\hat{r})(\vec{p}_2\cdot\hat{r})\right)\right)$$

$$= -\frac{1}{4\pi\varepsilon_0}\vec{\nabla}\left(\frac{1}{r^3}\right)\left(\vec{p}_1\cdot\vec{p}_2 - 3(\vec{p}_1\cdot\hat{r})(\vec{p}_2\cdot\hat{r})\right) + \frac{3}{4\pi\varepsilon_0}\vec{\nabla}\left((\vec{p}_1\cdot\hat{r})(\vec{p}_2\cdot\hat{r})\right)$$

- We have to do a little work here, but let's not get caught up in the intermediate steps:

$$\vec{\nabla}\left(\frac{1}{r^3}\right) = -\frac{3}{r^4}\,\hat{r}$$

Just vector calc

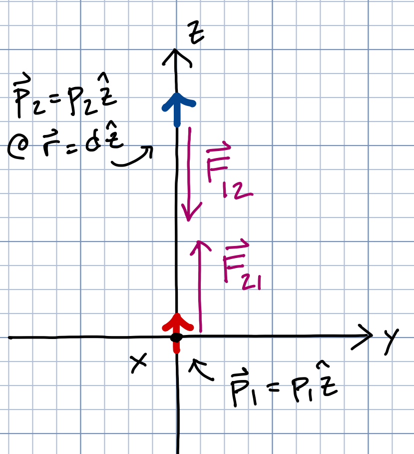
$$\vec{\nabla}(\vec{p}_1\cdot\hat{r}) = \vec{\nabla}\left(\frac{\vec{p}_1\cdot\vec{r}}{r}\right) = \frac{1}{r}\,\vec{p}_1 - (\vec{p}_1\cdot\hat{r})\frac{\vec{r}}{r^2}$$

$$\hookrightarrow \vec{F}_{12} = \frac{1}{4\pi\varepsilon_0}\times\frac{1}{r^4}\times\Big[3(\vec{p}_1\cdot\vec{p}_2)\hat{r} - 15(\vec{p}_1\cdot\hat{r})(\vec{p}_2\cdot\hat{r})\,\hat{r} + 3(\vec{p}_1\cdot\hat{r})\,\vec{p}_2 + 3(\vec{p}_2\cdot\hat{r})\,\vec{p}_1\Big]$$

- This force is more complicated than the point charge/dipole example. Remember that with our set-up, $\vec{r}$ is the separation vector from $\vec{p}_1$ to $\vec{p}_2$.

- Still, if we think of a generic dipole $\vec{p}$ as a little vector pointing from a negative charge to a positive charge, we can understand this force. Depending on the orientation of the dipoles & their relative position, the force can be attractive, repulsive, or something else entirely.

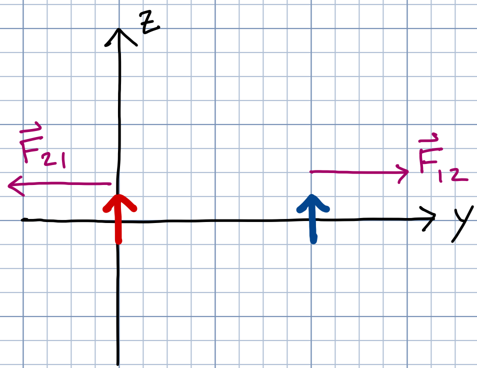
- Let's set up our axes so that $\hat{z} = \hat{p}_1$, and consider a few possibilities for $\vec{p}_2$ & $\vec{r}$:



$$\vec{F}_{12} = \frac{1}{4\pi\varepsilon_0}\frac{1}{d^4}\left(3\hat{z}\,p_1p_2 - 15\hat{z}\,p_1p_2 + 3p_1p_2\hat{z} + 3p_1p_2\hat{z}\right)$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{1}{d^4}(-6p_1p_2)\,\hat{z}$$

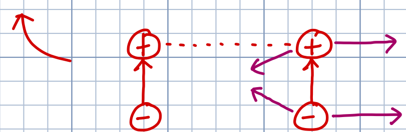
'Negative' end of $\vec{p}_2$ is slightly closer to positive end of $\vec{p}_1$, so there's a marginal attractive force. Flip dir. of $\vec{p}_2$ & the dir. of $\vec{F}_{12}$ reverses.



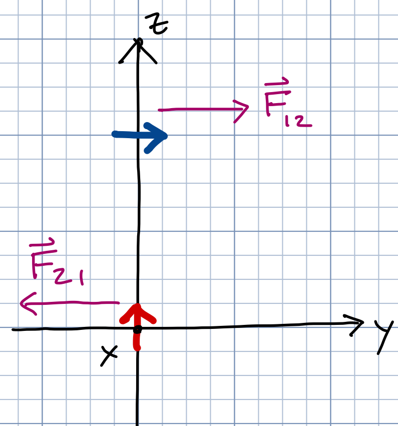
$\vec{p}_1 = p_1\hat{z}$ ⅋ $\vec{p}_2 = p_2\hat{z}$ @ $d\hat{y}$

$$\vec{F}_{12} = \frac{1}{4\pi\varepsilon_0}\frac{1}{d^4}\times\left(3\hat{y} - 0 + 0 + 0\right)p_1p_2$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{3p_1p_2}{d^4}\hat{y}$$



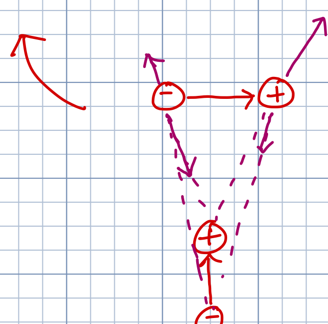
Net repulsive force



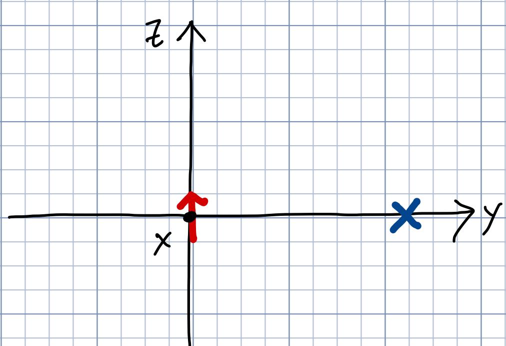
$\vec{p}_1 = p_1\hat{z}$ ⅋ $\vec{p}_2 = p_2\hat{y}$ @ $\vec{r} = d\hat{z}$

$$\hookrightarrow \vec{F}_{12} = \frac{1}{4\pi\varepsilon_0}\frac{p_1p_2}{d^4}\left(0 - 0 + 0 + 3\hat{y}\right)$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{3p_1p_2}{d^4}\hat{y}$$



Net force on $\vec{p}_2$ is in the $\hat{y}$ direction ⅋ is neither attractive or repulsive. Note that there is also a <u>torque</u> on $\vec{p}_2$!



$\vec{p}_1 = p_1\hat{z}$ ⅋ $\vec{p}_2 = p_2\hat{x}$ @ $\vec{r} = d\hat{y}$

$$\hookrightarrow \vec{F}_{12} = \frac{1}{4\pi\varepsilon_0}\frac{p_1p_2}{d^4}\times\left(0 - 0 + 0 + 0\right) = 0$$

Because $\vec{p}_1$, $\vec{p}_2$, ⅋ $\hat{r}$ are all orthogonal, $\vec{p}_2$ ⅋ $\vec{p}_1$ experience no force.

- So dipole-dipole forces are even weaker in that they fall off as $\sim 1/r^4$, and can point in a variety of directions (and even be <u>zero</u>) depending on the orientations and separation.

- Also, the dipoles may experience <u>torques</u> that try to turn them. The torques are a little more difficult to write out, but you can see <u>why</u> they occur

$$U = \frac{p_1 p_2}{4\pi\varepsilon_0 r^3} \times \Big(\underbrace{\hat{p}_1 \cdot \hat{p}_2 - 3(\hat{p}_1 \cdot \hat{r})(\hat{p}_2 \cdot \hat{r})} \Big)$$

Even @ a fixed distance b/t $\vec{p}_1$ & $\vec{p}_2$, the PE <u>decreases</u> if $\hat{p}_1$ and $\hat{p}_2$ turn to line up with $\hat{r}$.

- Higher-order multipoles experience even weaker forces w/ more complicated dependence on orientation and separation.